

High-rank ternary forms of even degree

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Abstract

We exhibit, for each positive even degree, a ternary form of rank strictly greater than the maximum rank of monomials. Together with an earlier result in the odd case, this gives a lower bound of

$$\left\lfloor \frac{d^2 + 2d + 5}{4} \right\rfloor,$$

for $d \geq 2$, on the maximum rank of degree d ternary forms with coefficients in an algebraically closed field of characteristic zero.

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1 Introduction

The Waring problem for the space of *all* forms of given degree and number of variables, say over the complex numbers, asks for the minimum number of summands that are required to write every such form as a sum of powers of linear forms. Since the minimum number for a given form is called its *Waring rank*, that problem can also be described as the determination of the maximum (Waring) rank in the mentioned space.

A similar problem for a *generic* form (of given degree and number of variables) deserved a particular attention in [5]. As reported in [3, Introduction], a non-obvious connection with a certain kind of interpolation problems has been recognized since the beginning of the twentieth century. According to Iarrobino [7], at the end Lazarsfeld noticed that the answer is provided, more specifically, by the recent Alexander-Hirschowitz interpolation theorem (established in [1]), a result which is now widely recognized as being outstanding.

Because of the similarity with the classical number-theoretic situation, the determination of the maximum rank in the space of all forms of given degree and number of variables has been named *little* Waring's (polynomial) problem in [6, p. 56]; while the *big* Waring's problem is the version about generic forms.

But, contrary to the classical situation, the big problem has been solved while the little one is still open (see [2] for details). Since polynomials over \mathbb{C} can be regarded as symmetric tensors, so that the Waring rank becomes the symmetric rank, these problems are also part of tensor theory. There are many similar basic questions in tensor theory that are still open, e.g., the symmetric Strassen conjecture, and some of them, e.g., determining the rank of matrix multiplication, are of utmost interest for applications.

A conjectural answer to the little Waring problem for ternary forms has been outlined at the end of [4, Introduction]. If that conjecture is true, then the lower bound given for odd degrees by [2, Th. 1] cannot be improved. For even degrees, one should similarly show that the greatest known rank (which to date is reached by monomials) can be raised by one. It is not unfrequent to encounter nontrivial differences between the even and the odd case in classical invariant theory and its developments. Indeed, traces of this phenomenon can be found since the classical works by Sylvester; for a recent example, see [9] in comparison with [8]. Here we work out the even case of the lower bound under consideration (Proposition 2.4).

2 A Promising Lower Bound

We pursue the ideas of [2] and borrow its basic framework, but restricted to the case of our interest. We consider an algebraically closed field \mathbb{K} of characteristic zero, the graded ring $S = \mathbb{K}[x, y, z]$ and its dual $T = \mathbb{K}[\alpha, \beta, \gamma]$ in the following sense: a perfect pairing $S_1 \times T_1 \rightarrow \mathbb{K}$ is understood, such that the (ordered) bases (x, y, z) and (α, β, γ) are dual to each other. The perfect pairing extends to *apolarity*, which may be described as the action of T on S such that $\sum a_{ijk} \alpha^i \beta^j \gamma^k$ acts as $\sum a_{ijk} \partial^{i+j+k} / \partial x^i \partial y^j \partial z^k$. Here we prefer the notation $\partial_\Theta F$ instead of $\Theta \cdot F$ for the action of $\Theta \in T$ on $F \in S$ (for instance, $\partial_\alpha F = \partial F / \partial x$). The *apolar algebra* A^F of a homogeneous $F \in S$ is the quotient of T over the (homogeneous) ideal $\{\Theta \in T : \partial_\Theta F = 0\}$. The (necessarily finite) dimension of A^F as a \mathbb{K} -vector space is called the *apolar length* of F and denoted by $\text{al}(F)$. Since A^F is a graded ring in a natural way, the apolar length is the sum of all values of its Hilbert function, which we denote by H^F . Let us also recall that if $F \in S_d$, then

$$H^F(d - n) = H^F(n) = \dim \{\partial_\Theta F : \Theta \in T_n\} .$$

Finally, let us recall from [2] the notation $H(n, d, s)$, which stands for the function on integers whose value at i is $\min \{\dim R_i, \dim R_{d-i}, s\}$, where R is the (graded) ring of polynomials in n variables over \mathbb{K} .

As in the approach of Buczyński and Teitler [2], two useful ingredients in the proof will be the following propositions.

Proposition 2.1. *In the above notation, if a form F has a power sum decomposition with all linear forms not annihilated by ∂_α , then it must contain at least $\text{al}(F) - \text{al}(\partial_\alpha F)$ summands.*

Proof. It follows from [2, Prop. 3]. □

Proposition 2.2. *In the above notation, for every power sum decomposition of F , at least $\text{al}(\partial_\alpha F) - \text{al}(\partial_{\alpha^2} F)$ of the linear forms are not annihilated by ∂_α .*

Proof. It follows from [2, Prop. 4]. \square

We shall also use the following elementary fact.

Proposition 2.3. *Let $R = \mathbb{K}[y, z]$, d a nonnegative integer, $r \in \{0, \dots, d\}$ and $F \in R_d$ a linear combination of r d -th powers of linear forms. Then*

1. *F belongs to $z^r R_{d-r}$ if and only if it is a scalar multiple of z^d ;*
2. *if the equivalent conditions in n.1 are true and the linear forms in the linear combination are pairwise independent, then there is at most one nonzero coefficient.*

Proof. When $r = d$, n.1 is trivial and to get n.2 it suffices to recall the following well known fact: given $d+1$ pairwise independent linear forms, their d -th powers are linearly independent.

Let us now assume $r < d$. Since the ‘if’ part in n.1 is trivial, let us assume that $F \in z^r R_{d-r}$. Let us suppose, in addition, that the linear forms are pairwise independent, and let us consider $F' := \partial^{d-r} F / \partial y^{d-r}$. All powers not proportional to z^d survive under derivation, but by the previous case they must occur with a zero coefficient in the derived combination, since $\deg F' = r$ (when $r = 0$ there is no summand). Therefore, even in the original combination there is at most one nonzero summand, proportional to z^d . Henceforth, F itself is proportional to z^d . To get n.1 even when the linear forms are not pairwise proportional, it suffices to reduce in the obvious way the linear combination to one with pairwise independent linear forms. \square

We are ready to prove our result: the existence, for each even degree $d > 0$, of a ternary forms whose rank exceeds the maximum rank of a ternary monomial (that is, $k^2 + 3k + 2$ when $d = 2k + 2$).

Proposition 2.4. *Let k be a nonnegative integer. There exists a form $F \in \mathbb{K}[x, y, z]$ of degree $2k + 2$ with $\text{rk } F > k^2 + 3k + 2$.*

Proof. For $k = 0$ it suffices to take a nondegenerate quadratic form. Hence we can assume $k \geq 1$. Let $S := \mathbb{K}[x, y, z]$, $T := \mathbb{K}[\alpha, \beta, \gamma]$ be the dual ring and $S^\alpha := \ker \partial_\alpha = \mathbb{K}[y, z]$. We show that

$$F := xy^{k-1}z^{k+2} + y^{2k}z^2$$

has rank strictly greater than $k^2 + 3k + 2$. Let $G := \partial_\alpha F = y^{k-1}z^{k+2}$ and let us calculate the Hilbert function H^G of the apolar algebra A^G . To this end, we recall that its value at n and at $2k + 1 - n$ is the dimension of the space $V_n := \{\partial_\Theta G : \Theta \in T_n\}$, which is spanned by the n -th partial derivatives of G . Therefore, for $0 \leq n \leq k - 1$ the space V_n is spanned by monomials

$$y^{k-1}z^{k+2-n}, \quad \dots, \quad y^{k-1-n}z^{k+2}$$

(in other terms, $V_n = y^{k-1-n} z^{k+2-n} S_n^\alpha$); for $k-1 \leq n \leq k+2$ it is spanned by

$$y^{k-1} z^{k+2-n}, \quad \dots, \quad y z^{2k-n}, \quad z^{2k+1-n}$$

(in other terms, $V_n = z^{k+2-n} S_{k-1}^\alpha$). Thus $H^G = H(2, 2k+1, k)$, hence

$$\text{al}(G) = \sum_n H^G(n) = 2 \left(\sum_{n=0}^{k-1} (n+1) + k \right) = k^2 + 3k.$$

We want to show that $\text{rk } F > k^2 + 3k + 2$. By contrary, let us assume $\text{rk } F \leq k^2 + 3k + 2$. Since $\partial_\alpha F = G$ and $\partial_{\alpha^2} F = 0$, by Proposition 2.2 we have that for every power sum decomposition of F , at least $k^2 + 3k$ of the involved linear forms are not annihilated by ∂_α . Let us take a decomposition as a sum of at most $k^2 + 3k + 2$ powers of pairwise independent linear forms. We can decompose F as $F^\alpha + B$, with F^α encompassing all summands with linear forms not annihilated by ∂_α , and B encompassing at most two summands, with linear forms in $S^\alpha = \mathbb{K}[y, z]$. Since $G = \partial_\alpha F = \partial_\alpha F^\alpha + \partial_\alpha B = \partial_\alpha F^\alpha$, according to Proposition 2.1, $\text{al}(F^\alpha) - \text{al}(G)$ cannot exceed the number of summands of whatever decomposition of F^α with all its linear forms not annihilated by ∂_α . Since F^α has such a decomposition with at most $k^2 + 3k + 2$ summands, it follows that $\text{al}(F^\alpha) \leq \text{al}(G) + k^2 + 3k + 2 = 2k^2 + 6k + 2$. We shall find a contradiction by showing that it must also be $\text{al}(F^\alpha) > 2k^2 + 6k + 2$. To this end, we make a direct computation of the Hilbert function H^{F^α} of A^{F^α} .

Since $\partial_{\alpha^2} F^\alpha = \partial_\alpha G = 0$, the space spanned by the n -th partial derivatives of F^α is the sum of the space V_{n-1} spanned by the $(n-1)$ -th partial derivatives of G (with $V_{-1} := \{0\}$ by convention) and the space W_n spanned by the n -th partial derivatives of F^α with respect to y, z . The space V_{n-1} has already been computed before for $0 \leq n-1 \leq k+2$.

Let us first consider the range $0 \leq n \leq k-1$. Since

$$F^\alpha - xy^{k-1} z^{k+2} = y^{2k} z^2 - B \in S^\alpha \quad (2.1)$$

and

$$xy^{k-1} z^{k+2-n}, \quad \dots, \quad xy^{k-1-n} z^{k+2}$$

are linearly independent modulo S^α , we have $V_{n-1} \cap W_n = \{0\}$ and $\dim W_n = n+1$. Hence $H^{F^\alpha}(n) = H^{F^\alpha}(2k+2-n) = \dim(V_{n-1} + W_n) = 2n+1$.

When $n = k$, the partial derivatives

$$\partial_{\gamma^n} F^\alpha, \quad \partial_{\beta \gamma^{n-1}} F^\alpha, \quad \dots, \quad \partial_{\beta^{n-1} \gamma} F^\alpha$$

are still linearly independent modulo S^α , because so are the monomials

$$xy^{k-1} z^2, \quad \dots, \quad xy z^k, \quad x z^{k+1}.$$

But now $\partial_{\beta^n} F^\alpha \in S^\alpha$, because of (2.1). However, we can check that it still lies outside $V_{n-1} = z^3 S_{k-1}^\alpha$. Suppose, on the contrary, that $\partial_{\beta^n} F^\alpha \in z^3 S_{k-1}^\alpha$. Since $\partial_{\beta^n} F^\alpha + \partial_{\beta^n} B = \partial_{\beta^n} F = \partial_{\beta^k} F$ is a scalar multiple of $y^k z^2$, we would have

$\partial_{\beta^n} B \in z^2 S_k^\alpha$. But B , and hence $\partial_{\beta^k} B$, can be written as a linear combination of two powers of independent linear forms (using zero coefficients if B has less than two summands). Therefore, by Proposition 2.3 we would have that $\partial_{\beta^k} B$ is a scalar multiple of $z^{k+2} \in z^3 S_{k-1}^\alpha$ (taking into account the initial assumption $k \geq 1$). Hence $\partial_{\beta^k} F$ would lie in $z^3 S_{k-1}^\alpha$ but, on the contrary, it is a nonzero scalar multiple of $y^k z^2$. We conclude that $\dim(V_{n-1} + W_n) = 2n + 1$ even when $n = k$.

Up till now we have shown that $H^{F^\alpha}(n) = H^{F^\alpha}(2k + 2 - n) = 2n + 1$ for $0 \leq n \leq k$, and we are checking that $\text{al}(F^\alpha) > 2k^2 + 6k + 2$. Hence we can conclude the proof by showing that the value of H^{F^α} at $n = k + 1$ is strictly greater than

$$2k^2 + 6k + 2 - 2 \sum_{n=0}^k (2n + 1) = 2k^2 + 6k + 2 - 2(k + 1)^2 = 2k.$$

As before, let us first note that since

$$xy^{k-1}z, \quad \dots, \quad xyz^{k-1}, \quad xz^k$$

are linearly independent modulo S^α , so are the partial derivatives $\partial_{\gamma^n} F^\alpha, \dots, \partial_{\beta^{n-3}\gamma^3} F^\alpha, \partial_{\beta^{n-2}\gamma^2} F^\alpha$ ($n = k + 1$). Since $\dim V_{n-1} = k$, this shows that $H^{F^\alpha}(n) \geq 2k$, and that we need only one more independent partial derivative. It will be enough to check that at least one of $\partial_{\beta^n} F^\alpha$ and $\partial_{\beta^{n-1}\gamma} F^\alpha$, which are both in S^α , is not in $V_{n-1} = z^2 S_{k-1}^\alpha$. If $\partial_{\beta^n} F^\alpha \in V_{n-1}$, since $\partial_{\beta^n} F^\alpha + \partial_{\beta^n} B = \partial_{\beta^n} F = \partial_{\beta^{k+1}} F \in z^2 S_{k-1}^\alpha = V_{n-1}$, by Proposition 2.3 we have that $\partial_{\beta^n} B$ is a scalar multiple of z^{k+1} . But in this case we have $\partial_{\beta^{n-1}} B = 0$ ($n - 1 = k \geq 1$), hence $\partial_{\beta^{n-1}\gamma} F^\alpha \notin z^2 S_{k-1}^\alpha = V_{n-1}$ because $\partial_{\beta^{n-1}\gamma} F^\alpha = \partial_{\beta^{n-1}\gamma} F^\alpha + \partial_{\beta^{n-1}\gamma} B = \partial_{\beta^{n-1}\gamma} F = \partial_{\beta^k\gamma} F$ is a nonzero scalar multiple of $y^k z$. \square

Let $r_{\max}(n, d)$ be the maximum rank of degree d forms in n variables. For $n = 3$ and $d \geq 2$, Proposition 2.4 and [2, Th. 1] (which holds for \mathbb{K} as well as for \mathbb{C} with no modifications in the proof) give the following lower bound.

Theorem 2.5. *For $d \geq 2$ we have*

$$r_{\max}(3, d) \geq \left\lfloor \frac{d^2 + 2d + 5}{4} \right\rfloor.$$

At the time of writing, we are inclined to believe that the opposite inequality has some chance of being true as well.

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